

ACI-matrices of constant rank over arbitrary fields ^{*†}

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Abstract

The columns of a $m \times n$ ACI-matrix over a field \mathbb{F} are independent affine subspaces of \mathbb{F}^m . An ACI-matrix has constant rank ρ if all its completions have rank ρ . Huang and Zhan (2011) characterized the $m \times n$ ACI-matrices of constant rank when $|\mathbb{F}| \geq \min\{m, n+1\}$. We complete their result characterizing the $m \times n$ ACI-matrices of constant rank over arbitrary fields. Quinlan and McTigue (2014) proved that every partial matrix of constant rank ρ has a $\rho \times \rho$ submatrix of constant rank ρ if and only if $|\mathbb{F}| \geq \rho$. We obtain an analogous result for ACI-matrices over arbitrary fields by introducing the concept of complete irreducibility.

1 Introduction

Let $\mathbb{F}[x_1, \dots, x_k]$ denote the set of polynomials in the indeterminates x_1, \dots, x_k with coefficients on a field \mathbb{F} . A matrix over $\mathbb{F}[x_1, \dots, x_k]$ is an **A**ffine **C**olumn **I**ndependent matrix or **ACI-matrix** if its entries are polynomials of degree at most one and no indeterminate appears in two different columns. A completion of an ACI-matrix is an assignment of values in \mathbb{F} to the indeterminates x_1, \dots, x_k . The ACI-matrices were introduced in 2010 by Brualdi, Huang and Zhan [3] as a generalization of **partial matrices** (matrices whose entries are either a constant or an indeterminate and with each indeterminate only appearing once). They proposed in [3, Problem 5] the problem of determining those $m \times n$ ACI-matrices such that the rank of any of its completions is equal to ρ with $0 \leq \rho \leq \min\{m, n\}$.

1.1 A geometric interpretation

Let us consider a collection \mathfrak{C} of $n+1$ affine subspaces of \mathbb{F}^m where \mathbb{F} is a field. If we choose one point of each one of the $n+1$ affine subspaces of \mathfrak{C} then the dimension of the affine subspace spanned by these $n+1$ points is an integer of the set $\{0, 1, \dots, \min\{m, n\}\}$. An interesting problem is to determine for any $\rho \in \{0, 1, \dots, \min\{m, n\}\}$ how are those collections $\mathfrak{C} = \{\mathcal{V}_0, \mathcal{V}_1, \dots, \mathcal{V}_n\}$ such that

$$\{\dim\langle P_0, P_1, \dots, P_n \rangle : P_i \in \mathcal{V}_i \text{ for } i = 0, 1, \dots, n\} = \{\rho\}.$$

As we will see below this question for the particular case in which \mathcal{V}_0 is the origin of \mathbb{F}^m coincides with the problem proposed by Brualdi, Huang and Zhan.

Let $\mathcal{V}_1, \dots, \mathcal{V}_n$ be n affine subspaces of \mathbb{F}^m . If \mathcal{V}_j has dimension d_j then it can be parametrized, with respect to the canonical base of \mathbb{F}^m , by

$$\begin{bmatrix} c_1^{(j)} \\ \vdots \\ c_m^{(j)} \end{bmatrix} + x_1^{(j)} \begin{bmatrix} a_{11}^{(j)} \\ \vdots \\ a_{m1}^{(j)} \end{bmatrix} + \dots + x_{d_j}^{(j)} \begin{bmatrix} a_{1d_j}^{(j)} \\ \vdots \\ a_{md_j}^{(j)} \end{bmatrix} = \begin{bmatrix} c_1^{(j)} + \sum_{k=1}^{d_j} a_{1k}^{(j)} x_k^{(j)} \\ \vdots \\ c_m^{(j)} + \sum_{k=1}^{d_j} a_{mk}^{(j)} x_k^{(j)} \end{bmatrix}.$$

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So, it seems quite natural to represent the collection $\{\mathcal{V}_1, \dots, \mathcal{V}_n\}$ by the $m \times n$ ACI-matrix

$$A = \begin{bmatrix} c_1^{(1)} + \sum_{k=1}^{d_1} a_{1k}^{(1)} x_k^{(1)} & \cdots & c_1^{(n)} + \sum_{k=1}^{d_n} a_{1k}^{(n)} x_k^{(n)} \\ \vdots & \ddots & \vdots \\ c_m^{(1)} + \sum_{k=1}^{d_1} a_{mk}^{(1)} x_k^{(1)} & \cdots & c_m^{(n)} + \sum_{k=1}^{d_n} a_{mk}^{(n)} x_k^{(n)} \end{bmatrix} \quad (1)$$

where the column j corresponds to the affine subspace \mathcal{V}_j .

A completion \hat{A} of the ACI-matrix A given in (1) is an assignment of values in \mathbb{F} to each one of the indeterminates

$$x_1^{(1)}, \dots, x_{d_1}^{(1)}; \dots; x_1^{(n)}, \dots, x_{d_n}^{(n)}.$$

Observe that the column j of \hat{A} corresponds to a point $P_j \in \mathcal{V}_j$. Therefore if $P_0 = (0, \dots, 0)$ is the origin of \mathbb{F}^m then

$$\text{rank}(\hat{A}) = \dim\langle \overrightarrow{P_0 P_1}, \dots, \overrightarrow{P_0 P_n} \rangle = \dim\langle P_0, P_1, \dots, P_n \rangle.$$

For any $\rho \in \{0, 1, \dots, \min\{m, n\}\}$ the problem of determining those collections $\{P_0, \mathcal{V}_1, \dots, \mathcal{V}_n\}$ of affine subspaces of \mathbb{F}^m such that $\dim\langle P_0, P_1, \dots, P_n \rangle = \rho$ for any choice of points $P_j \in \mathcal{V}_j$ for $j = 0, 1, \dots, n$ coincides with the problem of determining those $m \times n$ ACI-matrices over \mathbb{F} such that ρ is the rank of any of its completions.

1.2 The rank of an ACI-matrix

Definition 1.1. Let A be a $m \times n$ ACI-matrix over \mathbb{F} . The **rank of A** , $\text{rank}(A)$, is the set of integers that are the rank of some completion of A . The **Mrank of A** , $\text{Mrank}(A)$, is the highest rank of a completion of A , and the **mrank of A** , $\text{mrank}(A)$, is the lowest rank of a completion of A . We say that A has **constant rank ρ** if $\text{Mrank}(A) = \text{mrank}(A) = \rho$, that is, if $\text{rank}(A) = \{\rho\}$.

An **Affine Column** or **A-column** of size m is an ACI-matrix with one column and m rows. The ACI-matrices are described in terms of independent A-columns, where independent means that the A-columns share no variables. So $[C_1 \cdots C_n]$ is an $m \times n$ ACI-matrix if and only if C_1, \dots, C_n are independent A-columns of size m . The use of A-columns help us to introduce several concepts that appear when we consider ACI-matrices of constant rank.

Definition 1.2. Let $A = [C_1 \cdots C_n]$ be an $m \times n$ ACI-matrix over \mathbb{F} of constant rank ρ . We say that A is **full rank** if $\rho = \min\{m, n\}$. We distinguish three special types of full rank ACI-matrices:

- A is **square full rank** if $\rho = n = m$.
- A is **minimal full rank** if $\rho = m < n$ and for each $j \in \{1, \dots, n\}$ the $m \times (n - 1)$ ACI-matrix

$$[C_1 \cdots C_{j-1} C_{j+1} \cdots C_n]$$

is not full rank (i.e., it is not of constant rank $\rho = m \leq n - 1$).

- A is **maximal full rank** if $\rho = n < m$ and for each $v \in \mathbb{F}^m$ the $m \times (n + 1)$ ACI-matrix

$$[C_1 \cdots C_n v]$$

is not full rank (i.e., it is not of constant rank $\rho + 1 = n + 1 \leq m$).

It is important to keep in mind that if A is minimal full rank then it has less rows than columns, and that if A is maximal full rank then it has more rows than columns.

Example 1.3. In [2] we showed that there exist minimal and maximal full rank ACI-matrices over all finite fields. Namely, let $\mathbb{F}_q = \{f_1, \dots, f_q\}$ be the field with q elements:

(i) [2, Example 3.1] The following $2 \times (q + 1)$ ACI-matrix over \mathbb{F}_q is minimal full rank:

$$\begin{bmatrix} 1 + f_1 x_1 & \cdots & 1 + f_q x_q & x_{q+1} \\ x_1 & \cdots & x_q & 1 \end{bmatrix}.$$

(ii) [2, Proposition 2.1] The following $2q \times (q+1)$ ACI-matrix over \mathbb{F}_q is maximal full rank:

$$\left[\begin{array}{ccc|c} 1 & \cdots & 0 & x_{q+1} - f_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & x_{q+1} - f_q \\ \hline x_1 & \cdots & 0 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & x_q & 1 \end{array} \right].$$

On the other hand, in [2, Corollaries 2.1 and 3.1] we showed that minimal or maximal full rank ACI-matrices over infinite fields do not exist.

Example 1.4. (i) How to check that a given ACI-matrix is minimal full rank? Consider the 3×5 ACI-matrix over \mathbb{F}_2

$$A = \begin{bmatrix} 1 & y_2 & y_3 & 0 & 0 \\ 0 & 0 & y_3 & y_4 & 1 \\ y_1 & 1 & 1 & 1 & y_5 \end{bmatrix}.$$

Note that A is full rank since it has 5 variables and admits 2^5 different completions, all of them of rank 3. Moreover, for each $i = 1, 2, 3, 4, 5$ if we delete the i^{th} -column the resulting 3×4 ACI-matrix is not full rank since it admits a completion of rank 2. So A is minimal full rank.

(ii) Now we point at a sensitive property of the definition of maximal full rank ACI-matrices. The 5×3 ACI-matrix over \mathbb{F}_2

$$A = \begin{bmatrix} x_1 & 1 & 1 \\ 1 & 0 & x_4 \\ x_1 & 0 & x_4 \\ x_2 & 0 & 1 \\ 0 & x_3 & 1 \end{bmatrix}$$

has constant rank 3 since all its 2^4 completions have rank equal to 3. Let $\{e_1, e_2, e_3, e_4, e_5\}$ be the canonical base of the vectorial space \mathbb{F}_2^5 . It can be checked that

$$\text{rank}[A \ e_1] = \text{rank}[A \ e_2] = \text{rank}[A \ e_3] = \text{rank}[A \ e_4] = \text{rank}[A \ e_5] = \{3, 4\}.$$

None of these five augmented ACI-matrices is full rank. Does this imply that A is maximal full rank? No, since

$$\text{rank} \begin{bmatrix} x_1 & 1 & 1 & 1 \\ 1 & 0 & x_4 & 0 \\ x_1 & 0 & x_4 & 1 \\ x_2 & 0 & 1 & 1 \\ 0 & x_3 & 1 & 0 \end{bmatrix} = \{4\}.$$

In linear algebra it is usually enough to check a property for a basis to conclude that this property is true for all vectors. Although this is not the case when one wants to check that an ACI-matrix is maximal full rank.

1.3 Equivalent ACI-matrices

Assume that in the ACI-matrix $A = [C_1 \ \cdots \ C_n]$ the A-columns C_1, \dots, C_n are parametrized with respect to the canonical base of \mathbb{F}^m . If we consider a different base of \mathbb{F}^m then the parametrization of C_1, \dots, C_n with respect to this new base changes, although geometrically C_1, \dots, C_n do not change. This new parametrization is obtained by multiplying A from the left by a nonsingular constant matrix of order m . Note also that the order of the columns of A has no impact on its rank. These two observations motivate us to introduce in a natural way the terminology of equivalent ACI-matrices.

Definition 1.5. Two ACI-matrices A and B of the same size $m \times n$ are **equivalent**, $A \sim B$, if there exist a nonsingular constant T of order m and a permutation Q of order n such that $TAQ = B$.

The use of the permutation Q in the definition is not essential, but it is useful. It permits to reorganize the columns of an ACI-matrix so that its structure becomes more apparent.

Remark 1.6. *Given an $m \times n$ constant matrix A of rank ρ , it is well known that there exists a nonsingular constant T of order m such that TA is in row reduced echelon form. This is known as the Gauss elimination method. Moreover, there exists a permutation Q of order n such that*

$$TAQ = \left[\begin{array}{ccc|c} 1 & & 0 & * \\ & \ddots & & \\ 0 & & 1 & \\ \hline & 0 & & 0 \end{array} \right]$$

where we group together the ρ columns corresponding to the pivots in the first ρ columns. The blocks on the right do not appear if $\rho = n$ and the blocks on the bottom do not appear if $\rho = m$. So we have found a representative for the equivalence class of A with a simple structure. The equivalence for ACI-matrices of constant rank is, of course, an extension of the equivalence for constant matrices, and it is introduced with the idea of finding a representative with a simple structure that reveals its rank. Obviously the rank of an ACI-matrix is preserved by equivalence.

In our following result we will see that also minimality and maximality are preserved by equivalence.

Lemma 1.7. *Let A and B be equivalent ACI-matrices. We have that:*

- (i) *A is minimal full rank if and only if B is minimal full rank.*
- (ii) *A is maximal full rank if and only if B is maximal full rank.*

Proof. Let $m \times n$ be the size of A and B . As $A \sim B$ then there exist a nonsingular constant T of order m and a permutation Q of order n such that $B = TAQ$.

- (i) Assume that A is minimal full rank. Then B is full rank since

$$\begin{aligned} \text{rank}(B) &= \text{rank}(TAQ) = \{\text{rank}(T\hat{A}Q) : \hat{A} \text{ completion of } A\} \\ &= \{\text{rank}(\hat{A}) : \hat{A} \text{ completion of } A\} = \text{rank}(A) = \{m\}. \end{aligned}$$

Let us see now that B is minimal full rank. First we introduce some useful notation: $C_k(H)$ will denote the ACI-matrix obtained by deleting the column k of the ACI-matrix H .

As $B = TAQ$ then the columns of B are obtained by permuting the columns of TA . Let σ be a permutation of $\{1, \dots, n\}$ such that for each $j \in \{1, \dots, n\}$ the column j of B is equal to the column $\sigma(j)$ of TA . So B is minimal full rank since

$$\text{rank}(C_j(B)) = \text{rank}(C_{\sigma(j)}(TA)) = \text{rank}(TC_{\sigma(j)}(A)) = \text{rank}(C_{\sigma(j)}(A)) \neq \{m\}$$

where the last inequality follows from the fact that A is minimal full rank.

- (ii) Assume that A is maximal full rank. Then B is full rank since, as in item (i),

$$\text{rank}(B) = \text{rank}(TAQ) = \text{rank}(A) = \{n\}.$$

We conclude that B is maximal full rank since for each $v \in \mathbb{F}^m$ we have

$$\begin{aligned} \text{rank}\left(\begin{bmatrix} B & | & v \end{bmatrix}\right) &= \text{rank}\left(T^{-1}\begin{bmatrix} B & | & v \end{bmatrix}\right) = \text{rank}\left(\begin{bmatrix} T^{-1}B & | & T^{-1}v \end{bmatrix}\right) = \\ &= \text{rank}\left(\begin{bmatrix} AQ & | & T^{-1}v \end{bmatrix}\right) = \text{rank}\left(\begin{bmatrix} A & | & T^{-1}v \end{bmatrix}\right) \neq \{n+1\}. \end{aligned}$$

where the last inequality follows from the fact that A is maximal full rank.

□

2 ACI-matrices of constant rank over arbitrary fields

We start with a basic result that will be employed several times in this work.

Lemma 2.1. *Consider an ACI-matrix $\begin{bmatrix} A_{11} & A_{12} \\ 0_{r \times s} & A_{22} \end{bmatrix}$ where $r > m$, $s > n$ and $r + s \geq \max\{m, n\}$. The following two statements are equivalent:*

- (i) $\text{rank} \begin{bmatrix} A_{11} & A_{12} \\ 0_{r \times s} & A_{22} \end{bmatrix} = \{(m - r) + (n - s)\}$.
- (ii) $\text{rank}(A_{11}) = \{m - r\}$ and $\text{rank}(A_{22}) = \{n - s\}$.

Proof. Observe that A_{11} is $(m - r) \times s$ with $m - r \leq s$ and that A_{22} is $r \times (n - s)$ with $n - s \leq r$. Let $\begin{bmatrix} \widehat{A}_{11} & \widehat{A}_{12} \\ 0_{r \times s} & \widehat{A}_{22} \end{bmatrix}$ be any completion of $\begin{bmatrix} A_{11} & A_{12} \\ 0_{r \times s} & A_{22} \end{bmatrix}$.

(i) \Rightarrow (ii) We have that

$$(m - r) + (n - s) = \text{rank} \begin{bmatrix} \widehat{A}_{11} & \widehat{A}_{12} \\ 0_{r \times s} & \widehat{A}_{22} \end{bmatrix} \leq \text{rank}(\widehat{A}_{11}) + \text{rank}(\begin{bmatrix} \widehat{A}_{12} \\ \widehat{A}_{22} \end{bmatrix}) \leq (m - r) + (n - s)$$

then $\text{rank}(\widehat{A}_{11}) = m - r$ and so A_{11} has constant rank $m - r$.

$$(m - r) + (n - s) = \text{rank} \begin{bmatrix} \widehat{A}_{11} & \widehat{A}_{12} \\ 0_{r \times s} & \widehat{A}_{22} \end{bmatrix} \leq \text{rank} \begin{bmatrix} \widehat{A}_{11} & \widehat{A}_{12} \end{bmatrix} + \text{rank}(\widehat{A}_{22}) \leq (m - r) + (n - s)$$

then $\text{rank}(\widehat{A}_{22}) = n - s$ and A_{22} has constant rank $n - s$.

(ii) \Rightarrow (i) The rank of \widehat{A}_{22} is equal to the number of columns of $\begin{bmatrix} \widehat{A}_{12} \\ \widehat{A}_{22} \end{bmatrix}$. Then we have

$$\text{rank} \begin{bmatrix} \widehat{A}_{11} & \widehat{A}_{12} \\ 0_{r \times s} & \widehat{A}_{22} \end{bmatrix} = \text{rank} \begin{bmatrix} \widehat{A}_{11} & 0 \\ 0_{r \times s} & \widehat{A}_{22} \end{bmatrix} = \text{rank}(\widehat{A}_{11}) + \text{rank}(\widehat{A}_{22}) = (m - r) + (n - s).$$

And so $\text{rank} \begin{bmatrix} A_{11} & A_{12} \\ 0_{r \times s} & A_{22} \end{bmatrix} = \{(m - r) + (n - s)\}$.

□

For a better understanding of the structure of the constant rank ACI-matrices we will make use of the following result of Brualdi, Huang and Zhan [3, Theorem 3].

Theorem 2.2. ([3]) *Let A be an $m \times n$ ACI-matrix over an arbitrary field \mathbb{F} and let ρ be an integer such that $1 \leq \rho < \min\{m, n\}$. The following two statements are equivalent:*

- (i) $\text{Mrank}(A) \leq \rho$.
- (ii) *For some positive integers r and s with $\rho = (m - r) + (n - s)$ there exist a nonsingular constant T of order m and a permutation Q of order n such that $TAQ = \begin{bmatrix} A_{11} & A_{12} \\ 0_{r \times s} & A_{22} \end{bmatrix}$. The upper blocks A_{11} and A_{12} do not appear if $r = m$ and the right blocks A_{21} and A_{22} do not appear if $s = n$.*

Observe that $\rho = (m - r) + (n - s)$ and $\rho < \min\{m, n\}$ implies that

$$r + s = m + n - \rho > \max\{m, n\}$$

So $m - r < s$ and $n - s < r$. Therefore, in part (ii) of Theorem 2.2, A_{11} has less rows than columns and A_{22} has less columns than rows. With all this in mind, an immediate consequence of Theorem 2.2 and Lemma 2.1 is the following result for ACI-matrices of constant rank.

Corollary 2.3. *Let A be an $m \times n$ ACI-matrix over an arbitrary field \mathbb{F} and let ρ be an integer such that $1 \leq \rho < \min\{m, n\}$. The following two statements are equivalent:*

(i) $\text{rank}(A) = \{\rho\}$.

(ii) *For some positive integers r and s with $\rho = (m - r) + (n - s)$ there exist a nonsingular constant T of order m and a permutation Q of order n such that $TAQ = \begin{bmatrix} A_{11} & A_{12} \\ 0_{r \times s} & A_{22} \end{bmatrix}$. The upper blocks A_{11} and A_{12} do not appear if $r = m$ and the right blocks A_{21} and A_{22} do not appear if $s = n$. Moreover, if $r < m$ then $\text{rank}(A_{11}) = \{m - r\}$ and if $s < n$ then $\text{rank}(A_{22}) = \{n - s\}$.*

2.1 Sufficient and necessary condition for ACI-matrices of constant rank

Huang and Zhan in [4, Theorem 5] characterized the $m \times n$ ACI-matrices of constant rank over a field \mathbb{F} with $|\mathbb{F}| \geq \max\{m, n + 1\}$.

Theorem 2.4. ([4]) *Let A be a $m \times n$ ACI-matrix over a field \mathbb{F} with $|\mathbb{F}| \geq \max\{m, n + 1\}$. Then A has constant rank ρ if and only if*

$$A \sim \begin{bmatrix} B & * & * \\ 0 & 0 & * \\ 0 & 0 & C \end{bmatrix} \quad (2)$$

for some ACI-matrices B and C which are square upper triangular with nonzero constant diagonal entries and whose orders sum to ρ .

We remark that in (2) some block rows or/and block columns may be void. Now in the next theorem we will rewrite Theorem 2.4 making these degenerate cases more explicit by dividing the result in different cases depending on the relation of m and n with ρ . We will use square upper triangular ACI-matrices with all its diagonal entries equal to 1 instead of square upper triangular ACI-matrices with nonzero constant diagonal entries. It is clear that this change can be done.

Theorem 2.4 (detailed version). *Let A be a $m \times n$ ACI-matrix of constant rank ρ with $1 \leq \rho \leq \min\{m, n\}$ over a field \mathbb{F} with $|\mathbb{F}| \geq \max\{m, n + 1\}$. Depending on m , n and ρ we have the following possibilities:*

(i) $\rho = m = n$ if and only if $A \sim \begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$.

(ii) $\rho = m < n$ if and only if $A \sim \left[\begin{array}{ccc|c} 1 & & * & \\ & \ddots & & \\ 0 & & 1 & * \end{array} \right]$.

(iii) $\rho = n < m$ if and only if $A \sim \left[\begin{array}{c|cc} * & & \\ \hline 1 & & * \\ & \ddots & \\ 0 & & 1 \end{array} \right]$.

(iv) $1 \leq \rho < \min\{m, n\}$ if and only if for some positive integers r and s with $r + s = m + n - \rho$

$$A \sim \left[\begin{array}{ccc|c|c} 1 & & * & & \\ & \ddots & & * & * \\ 0 & & 1 & & \\ \hline & & & 0_{r \times s} & * \\ & & & & \hline & & & 1 & * \\ & & & & \ddots \\ & & & 0 & 1 \end{array} \right] \quad (3)$$

where the upper blocks do not appear if $r = m$ and the right blocks do not appear if $s = n$.

Remark. Note that if \mathbb{F} is an infinite field then $|\mathbb{F}| \geq \max\{m, n+1\}$ and items (i) to (iv) are satisfied. In [1] we proved that item (i) is true for any field \mathbb{F} without the restriction $|\mathbb{F}| \geq \max\{m, n+1\}$. In [2, Lemma 2.1 and Lemma 3.1] we proved the existence of minimal full rank ACI-matrices and of maximal full rank ACI-matrices over all finite fields (see Example 1.3), and we showed that if A is minimal full rank and B is maximal full rank then

$$A \not\sim \left[\begin{array}{ccc|c} 1 & & * & \\ & \ddots & & \\ 0 & & 1 & * \end{array} \right] \quad \text{and} \quad B \not\sim \left[\begin{array}{c|c} & * \\ \hline 1 & * \\ & \ddots \\ 0 & 1 \end{array} \right].$$

Therefore the results of items (ii) and (iii) of Theorem 2.4 can not be extended to finite fields \mathbb{F} disregarding completely the restriction on $|\mathbb{F}|$. Moreover, in [2, Theorem 4.1 and Theorem 4.2] we characterized the full rank ACI-matrices over arbitrary fields with the help of the minimal and the maximal full rank ACI-matrices as can be seen in items (ii) and (iii) of Theorem 2.5 below.

It is worthy to mention the different approach taken by McTigue and Quinlan (see [8] and [5, Corollary 6.1]). They proved that if a partial matrix P with constant rank ρ is equivalent to an ACI-matrix of the types found in (i), (ii), (iii) and (iv) of Theorem 2.4, then P has a $\rho \times \rho$ submatrix of constant rank ρ . On the other hand, for any prime power q they constructed a $(q+1) \times 2q$ partial matrix A_q over \mathbb{F}_q that has constant rank $q+1$ and has no $(q+1) \times (2q-1)$ submatrix of constant rank $q+1$ (for instance $A_2 = \begin{bmatrix} 1 & 1 & x_2 & 0 \\ 1 & 0 & 0 & x_3 \\ 1 & x_1 & 1 & 1 \end{bmatrix}$). So A_q has no $(q+1) \times (q+1)$ submatrix of constant rank $q+1$ and Theorem 2.4 can not be extended to \mathbb{F}_q disregarding completely the restriction on $|\mathbb{F}_q|$. It is important to realize that the mentioned conditions for A_q imply that A_q is minimal full rank and so, interestingly, both approaches have led us to the same type of matrices.

One of our main objectives in this work is to complete the characterization of the $m \times n$ ACI-matrices of constant rank ρ over arbitrary fields. This is done in our next result that includes the case when $\rho < \min\{m, n\}$

Theorem 2.5. *Let A be a $m \times n$ ACI-matrix over an arbitrary field \mathbb{F} . Then A has constant rank ρ if and only if*

$$A \sim \left[\begin{array}{ccc} B & * & * \\ 0 & 0 & * \\ 0 & 0 & C \end{array} \right] \quad (4)$$

for some ACI-matrices B and C such that B is square upper triangular with nonzero constant diagonal entries or is minimal full rank; C is square upper triangular with nonzero constant diagonal entries or is maximal full rank; and the number of rows of B plus the number of columns of C is ρ .

We again remark that in (4) some block rows or/and block columns may be void. In the next theorem we will rewrite Theorem 2.5 making these degenerate cases more explicit by dividing the result in different cases depending on the relation of m and n with ρ . This will facilitate the proof of the result. And we will use square upper triangular ACI-matrices with all its diagonal entries equal to 1 instead of square upper triangular ACI-matrices with nonzero constant diagonal entries.

Theorem 2.5 (detailed version). *Let A be a $m \times n$ ACI-matrix of constant rank ρ with $1 \leq \rho \leq \min\{m, n\}$ over an arbitrary field \mathbb{F} . Depending on m , n and ρ we have the following possibilities:*

$$(i) \quad \rho = m = n \text{ if and only if } A \sim \left[\begin{array}{ccc} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{array} \right].$$

$$(ii) \quad \rho = m < n \text{ if and only if } A \sim [B \quad *] \text{ where either } B = \left[\begin{array}{ccc} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{array} \right] \text{ or } B \text{ is } m \times n' \text{ minimal full rank with } m < n' \leq n.$$

$$(iii) \quad \rho = n < m \text{ if and only if } A \sim \left[\begin{array}{c} * \\ C \end{array} \right] \text{ where either } C = \left[\begin{array}{ccc} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{array} \right] \text{ or } C \text{ is } m' \times n \text{ maximal full rank with } n < m' \leq m.$$

(iv) $\rho < \min\{m, n\}$ if and only if one of the following possibilities is satisfied:

(a) there exist positive integers $r < m$ and $s < n$ with $\rho = (m - r) + (n - s)$ such that

$$A \sim \left[\begin{array}{c|c} B & * \\ \hline 0_{r \times s} & C \end{array} \right]$$

where $[B \quad *]$ has less rows than columns with either $B = \begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$ or B is minimal full rank, and $\begin{bmatrix} * \\ C \end{bmatrix}$ has more rows than columns with either $C = \begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$ or C is maximal full rank.

(b) there exists a positive integer $r < m$ with $\rho = m - r$ such that $A \sim \left[\begin{array}{c|c} B & * \\ \hline 0_{r \times n} & \end{array} \right]$ where $[B \quad *]$ has less rows than columns with either $B = \begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$ or B is minimal full rank.

(c) there exists a positive integer $s < n$ with $\rho = n - s$ such that $A \sim \left[\begin{array}{c|c} 0_{m \times s} & * \\ \hline & C \end{array} \right]$ where $\begin{bmatrix} * \\ C \end{bmatrix}$ has more rows than columns with either $C = \begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$ or C is maximal full rank.

Proof. Item (i) was proved in [1, Theorem 3.1] and items (ii) and (iii) were proved in [2, Theorems 4.1 and 4.2]. Let us prove item (iv):

\Rightarrow) By Corollary 2.3, for some positive integers r and s with $\rho = (m - r) + (n - s)$ there exist a nonsingular constant T of order m and a permutation matrix Q of order n such that

$$TAQ = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline 0_{r \times s} & A_{22} \end{array} \right]$$

where the upper blocks do not appear if $r = m$ and the right blocks do not appear if $s = n$.

Corollary 2.3 asserts that if $r < m$ then A_{11} is $(m - r) \times s$ with $\text{rank}(A_{11}) = \{m - r\}$. As $m - r = \rho - n + s < s$ then by item (ii) of this theorem, there exist a nonsingular constant T_1 of order $m - r$ and a permutation matrix Q_1 of order s such that $T_1 A_{11} Q_1 = [B \quad *]$ where either $B = \begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$ or B is a $(m - r) \times s'$ minimal full rank with $m - r < s' \leq s$.

Corollary 2.3 also asserts that if $s < n$ then A_{22} is $r \times (n - s)$ with $\text{rank}(A_{22}) = \{n - s\}$. As $n - s = \rho - m + r < r$ then by item (iii) of this theorem, there exist a nonsingular constant T_2 of order r and a permutation matrix Q_2 of order $n - s$ such that $T_2 A_{22} Q_2 = \begin{bmatrix} * \\ C \end{bmatrix}$ where either $C = \begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$ or C is a $r' \times (n - s)$ maximal full rank with $n - s < r' \leq r$.

Observe that $r = m$ and $s = n$ is not possible. So we consider three cases:

(a) if $r < m$ and $s < n$ then

$$A \sim TAQ \sim \left[\begin{array}{c|c} T_1 & 0 \\ \hline 0 & T_2 \end{array} \right] TAQ \left[\begin{array}{c|c} Q_1 & 0 \\ \hline 0 & Q_2 \end{array} \right] = \left[\begin{array}{c|c} T_1 A_{11} Q_1 & * \\ \hline 0_{r \times s} & T_2 A_{22} Q_2 \end{array} \right] = \left[\begin{array}{c|c} B & * \\ \hline 0_{r \times s} & C \end{array} \right].$$

(b) if $r < m$ and $s = n$ then

$$A \sim \left[\begin{array}{c|c} T_1 & 0 \\ \hline 0 & I_r \end{array} \right] (TAQ) Q_1 = \left[\begin{array}{c|c} T_1 & 0 \\ \hline 0 & I_r \end{array} \right] \left[\begin{array}{c} A_{11} \\ 0_{r \times n} \end{array} \right] Q_1 = \left[\begin{array}{c|c} T_1 A_{11} Q_1 \\ \hline 0_{r \times n} \end{array} \right] = \left[\begin{array}{c|c} B & * \\ \hline 0_{r \times n} & \end{array} \right].$$

(c) if $r = m$ and $s < n$ then

$$A \sim T_2 (TAQ) \left[\begin{array}{c|c} I_s & 0 \\ \hline 0 & Q_2 \end{array} \right] = T_2 \left[\begin{array}{c|c} 0_{m \times s} & A_{22} \end{array} \right] \left[\begin{array}{c|c} I_s & 0 \\ \hline 0 & Q_2 \end{array} \right] = \left[\begin{array}{c|c} 0_{m \times s} & T_2 A_{22} Q_2 \end{array} \right] = \left[\begin{array}{c|c} 0_{m \times s} & * \\ \hline & C \end{array} \right].$$

\Leftrightarrow) We will prove for each one of the three cases that $\rho < \min\{m, n\}$:

(a) By hypothesis $A \sim \left[\begin{array}{c|c} B & * \\ \hline 0_{r \times s} & C \end{array} \right]$. As $[B \quad *]$ has less rows than columns then $m - r < s$ and so

$$\rho = (m - r) + (n - s) < s + (n - s) = n.$$

On the other hand, as $\begin{bmatrix} * \\ C \end{bmatrix}$ has more rows than columns then $r > n - s$ and so

$$\rho = (m - r) + (n - s) < (m - r) + r = m.$$

(b) By hypothesis $A \sim \left[\begin{array}{c|c} B & * \\ \hline 0_{r \times n} & \end{array} \right]$. As $[B \quad *]$ has less rows than columns then $m - r < n$, so $\rho = m - r < \min\{m, n\}$.

(c) By hypothesis $A \sim \left[\begin{array}{c|c} 0_{m \times s} & * \\ \hline C & \end{array} \right]$. As $\begin{bmatrix} * \\ C \end{bmatrix}$ has more rows than columns then $n - s < m$, so $\rho = n - s < \min\{m, n\}$.

□

Note 2.6. If A is a constant matrix then A is equivalent to one of the following constant matrices:

$$\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}, \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \left| \begin{array}{c} * \\ * \\ * \end{array} \right., \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}, \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \left| \begin{array}{c} * \\ * \\ 0 \end{array} \right.$$

which are in row reduced echelon form and correspond, respectively, to the patterns given in items (i), (ii), (iv)(b) and (iv)(a) of Theorem 2.5. So, in a sense Theorems 2.4 and 2.5 provide a generalization of the row reduced echelon form (extended by columns permutation) for ACI-matrices.

2.2 An example

Consider the 7×7 ACI-matrix over \mathbb{F}_2

$$A = \begin{bmatrix} x_1 + y_1 & x_2 + 1 & 1 & x_4 & 0 & x_6 + 1 & x_7 \\ x_1 & 1 & 1 & x_4 & 1 & x_6 + 1 & x_7 \\ x_1 & x_2 + 1 & 0 & 0 & x_5 & 0 & 0 \\ y_1 + 1 & y_2 & x_3 & y_4 & 0 & x_6 & 1 \\ 0 & x_2 + y_2 + 1 & x_3 & y_4 & 0 & x_6 & 1 \\ 1 & x_2 & 1 & x_4 & 0 & x_6 + 1 & x_7 \\ y_1 & x_2 + y_2 + 1 & x_3 & y_4 & x_5 & x_6 & 1 \end{bmatrix}$$

It has 10 variables and each variable can take 2 values. With a computer it is easy to calculate the rank of the 2^{10} different completions of A to conclude that A has constant rank $\rho = 5$. Our intention is to find an ACI-matrix equivalent to A which is expressed as equation (4) given in Theorem 2.5.

Consider any variable of A , for instance x_1 . Permute rows and columns of A so that x_1 is placed in the (1,1)-position. In this case no permutation of rows or columns is necessary. After that delete x_1 from the rest of entries of the first column by multiplying by the left by an adequate nonsingular ACI-matrix T . So variable x_1 only appears in the (1,1)-entry of the ACI-matrix $A_1 = TA$ with $A_1 \sim A$. Consider now any variable that is neither in the first row nor in the first column of A_1 , for instance x_2 . We can proceed as before so that we will obtain $A_2 \sim A_1$ such that x_1 only appears on the (1,1)-position of A_2 and x_2 only appears on the (2,2)-position of A_2 . Repeat this process with any variable that is neither in the first two rows nor in the first two columns of A_2 . And so on until no

variable remains. At the end of this procedure we obtain

$$A \sim A_5 = \left[\begin{array}{c|cccccc} \textcircled{x_1} + y_1 + 1 & 1 & \mathbf{0} & \mathbf{0} & 0 & \mathbf{0} & \mathbf{0} \\ y_1 & \textcircled{x_2} & \mathbf{0} & \mathbf{0} & 1 & \mathbf{0} & \mathbf{0} \\ y_1 + 1 & y_2 & \textcircled{x_3} & y_4 & 0 & x_6 & 1 \\ y_1 + 1 & 0 & 1 & \textcircled{x_4} & 1 & x_6 + 1 & x_7 \\ 1 & 0 & \mathbf{0} & \mathbf{0} & \textcircled{x_5} + 1 & \mathbf{0} & \mathbf{0} \\ 1 & 1 & \mathbf{0} & \mathbf{0} & 1 & \mathbf{0} & \mathbf{0} \\ y_1 & 1 & \mathbf{0} & \mathbf{0} & 0 & \mathbf{0} & \mathbf{0} \end{array} \right] \quad (5)$$

where we have circled the chosen variables x_1, x_2, x_3, x_4, x_5 , which we will call *pivots*.

By Corollary 2.3 we know that there exists positive integers r and s with

$$r + s = 7 + 7 - 5 = 9$$

such that

$$A_5 \sim \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline 0_{r \times s} & A_{22} \end{array} \right].$$

It is clear that this equivalence can be realized through a permutation of rows and columns¹. This is not a coincidence. Although it is not necessary, we will prove it *formally* so that it is possible to extrapolate the arguments whenever the number of pivots and the constant rank are equal.

Let x_{i_1}, \dots, x_{i_h} be the pivots that appear in A_{11} . Proceeding with A_{11} as we did with A we obtain that $A_{11} \sim T_1 A_{11} Q_1 = A'_{11}$ with x_{i_v} only appearing in the (v, v) -position of A'_{11} . In the same way, let x_{j_1}, \dots, x_{j_k} the pivots that appear in A_{22} . Again, proceeding with A_{22} as we did with A we obtain $A_{22} \sim T_2 A_{22} Q_2 = A'_{22}$ with x_{j_w} only appearing in the (w, w) -position of A'_{22} . So

$$\left[\begin{array}{c|c} T_1 & 0 \\ \hline 0 & T_2 \end{array} \right] \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline 0_{r \times s} & A_{22} \end{array} \right] \left[\begin{array}{c|c} Q_1 & 0 \\ \hline 0 & Q_2 \end{array} \right] = \left[\begin{array}{c|c} A'_{11} & T_1 A_{12} Q_2 \\ \hline 0_{r \times s} & A'_{22} \end{array} \right].$$

Finally, if a row of $T_1 A_{12} Q_2$ contains the pivot x_{j_t} for some $t = 1, \dots, k$ then we can delete it by adding a multiple of the t -th row of A'_{22} . Let A'_{12} be the ACI-matrix obtained after we have deleted the pivots x_{j_1}, \dots, x_{j_k} of $T_1 A_{12} Q_2$. Then

$$\left[\begin{array}{c|c} A'_{11} & T_1 A_{12} Q_2 \\ \hline 0_{r \times s} & A'_{22} \end{array} \right] \sim \left[\begin{array}{c|c} A'_{11} & A'_{12} \\ \hline 0_{r \times s} & A'_{22} \end{array} \right] = A'.$$

In A'_{12} there will remain $l = 5 - h - k$ pivots that will appear in l rows of A'_{12} that are different from its first h rows and in l columns of A'_{12} that are different from its first k columns. As A'_{11} has $7 - r$ rows and A'_{22} has $7 - s$ columns then

$$5 = (7 - r) + (7 - s) \geq (h + l) + (k + l) = 5 + l$$

and so $l = 0$, $h = 7 - r$ and $k = 7 - s$. The 5 pivots in A' are in the $(1, 1), \dots, (7 - r, 7 - r)$ positions of A'_{11} and in the $(1, 1), \dots, (7 - s, 7 - s)$ positions of A'_{22} .

Note that the pivots are in the first 5 rows of A' . So each of the first 5 rows of A_5 only participates in the row of A' in which the same pivot appears. In other words, there exists a permutation P of order 5 such that

$$A' = \left[\begin{array}{cc} P & T_{12} \\ 0_{2 \times 5} & T_{22} \end{array} \right] A_5 Q \quad \text{with } \det(T_{22}) \neq 0$$

The position of the pivots of A' does not depend of T_{12} and T_{22} . On the other hand, $r + s = 9$ implies that $r \geq 2$ and so the $0_{r \times s}$ block of A' does not depend of T_{12} and T_{22} . So without loss of generality we can assume the simplest situation: $T_{12} = 0_{5 \times 2}$ and $T_{22} = I_2$. Therefore A' can be obtained by

¹In (5) we have marked the zeros in bold face so that, when reordered, give the zero block $0_{r \times s}$ of (6)

permuting the columns of one ACI-matrix which in turn is obtained by permuting the first 5 rows of A_5 . This means that the zeros of the zero block we are looking for should already appear in (5). It should be straightforward to find such a block of zeros.

$$A' = \left[\begin{array}{c|c} A'_{11} & A'_{12} \\ \hline 0_{5 \times 4} & A'_{22} \end{array} \right] = \left[\begin{array}{cccc|ccc} \textcircled{x_3} & y_4 & x_6 & 1 & y_1 + 1 & y_2 & 0 \\ 1 & \textcircled{x_4} & x_6 + 1 & x_7 & y_1 + 1 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & \textcircled{x_1} + y_1 + 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & y_1 & \textcircled{x_2} & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & \textcircled{x_5} + 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & y_1 & 1 & 0 \end{array} \right] \quad (6)$$

This procedure for finding the block of zeros in some ACI-matrix equivalent to A , can be applied to other examples whenever we are given a ρ constant rank ACI-matrix for which we can find ρ pivots. In the present example A has constant rank $\rho = 5$ and we have found 5 pivots.

Now we continue our search of an equivalent ACI-matrix of A that is of type (4) of Theorem 2.5:

- By checking all completions of A'_{11} we know that A'_{11} has constant rank 2. Moreover, as A'_{11} has no constant column then $A'_{11} \not\sim \left[\begin{array}{cc|cc} 1 & * & * & * \\ 0 & 1 & * & * \end{array} \right]$. So, by Theorem 2.5 (ii), $A'_{11} \sim [B \quad *]$ where B is minimal full rank. Note that if $x_4 = y_4 = 0$ then the second column of A'_{11} is null, so this column can not be part of a minimal full rank ACI-matrix. Moreover, in Example 1.3 (i) we saw that $\left[\begin{array}{ccc} x_3 & x_6 & 1 \\ 1 & x_6 + 1 & x_7 \end{array} \right]$ is minimal full rank. Therefore

$$A'_{11} \sim [B \quad *] = \left[\begin{array}{ccc|c} x_3 & x_6 & 1 & y_4 \\ 1 & x_6 + 1 & x_7 & x_4 \end{array} \right]$$

- On the other hand, by checking all completions of A'_{22} we known that A'_{22} has constant rank 3. Moreover, $A'_{22} \not\sim \left[\begin{array}{ccc} * & * & * \\ * & * & * \\ \hline 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{array} \right]$ since no linear combination of the rows of A'_{22} is equal to either $[1 \ 0 \ 0]$ or $[0 \ 1 \ 0]$ or $[0 \ 0 \ 1]$. So, by Theorem 2.5 (iii), $A'_{22} \sim \left[\begin{array}{c} * \\ C \end{array} \right]$ where C is maximal full rank. We want to know if A'_{22} is maximal full rank, so we proceed to check if some augmented ACI-matrix $[A'_{22} \mid v]$ with $v \in \mathbb{F}_2^5$ has constant rank. We discover that in fact

$$\text{rank} \left[\begin{array}{ccc|c} x_1 + y_1 + 1 & 1 & 0 & 1 \\ y_1 & x_2 & 1 & 0 \\ 1 & 0 & x_5 + 1 & 0 \\ 1 & 1 & 1 & 0 \\ y_1 & 1 & 0 & 0 \end{array} \right] = \{4\} \quad (7)$$

and therefore A'_{22} is not maximal full rank. From Equation (7) it follows that deleting the first row of A'_{22} we obtain an ACI-matrix, that we will denote C , such that

$$\text{rank}(C) = \text{rank} \left[\begin{array}{ccc} y_1 & x_2 & 1 \\ 1 & 0 & x_5 + 1 \\ 1 & 1 & 1 \\ y_1 & 1 & 0 \end{array} \right] = \{3\}$$

Again we proceed to check if some augmented ACI-matrix $[C \mid w]$ with $w \in \mathbb{F}_2^4$ has constant rank 4. As this is not the case then C is maximal full rank.

Therefore we conclude that

$$A \sim \left[\begin{array}{c|c|c} B & * & * \\ \hline 0 & 0 & * \\ \hline 0 & 0 & C \end{array} \right] \sim \left[\begin{array}{ccc|c|ccc} \textcircled{x_3} & x_6 & 1 & y_4 & y_1 + 1 & y_2 & 0 \\ 1 & x_6 + 1 & x_7 & \textcircled{x_4} & y_1 + 1 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & \textcircled{x_1} + y_1 + 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & y_1 & \textcircled{x_2} & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & \textcircled{x_5} + 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & y_1 & 1 & 0 \end{array} \right]$$

where B is minimal full rank and C is maximal full rank, a possibility that Theorem 2.5 considers.

Remark 2.7. *In order to check that A has constant rank, we can avoid checking the rank of all its completions ($2^{10} = 1024$ as we pointed out in the beginning of the subsection) and instead check the rank of two much smaller ACI-matrices with less variables. We proceed as follows. We take any completion, for instance the one obtained by assigning 0 to all variables, which will give us a rank of 5. So if $\text{rank}(A) = \{\rho\}$ then $\rho = 5$. We assume that A has constant rank 5 and we proceed in the same way that we did in this subsection. At some point we prove that A'_{11} has constant rank 2 and A'_{22} has constant rank 3. We conclude, from Lemma 2.1, that $\text{rank}(A) = \text{rank}(A') = \{5\}$. Note that for calculating the ranks of A'_{11} and of A'_{22} we calculate the rank of all their completions. But the number of completions of A'_{11} is $2^5 = 32$ and the number of completion of A'_{22} is $2^4 = 16$. These numbers are much smaller than 1024. Moreover, the size of A'_{11} and A'_{22} are smaller than the size of A .*

3 The concept of reducibility for ACI-matrices

If A is a $m \times n$ constant matrix of rank ρ then it is well known that we can delete $m - \rho$ rows and $n - \rho$ columns in such a way that the $\rho \times \rho$ submatrix of A that we obtain has rank ρ . We would like to know if ACI-matrices share this property. First we will consider partial matrices. We might naively expect that any partial matrix with constant rank ρ must also have a $\rho \times \rho$ submatrix of constant rank ρ . McTigue and Quinlan studied the rank of partial matrices in [6, 7, 8] and proved that this is not the case.

Theorem 3.1. ([8]) *Every partial matrix A of constant rank ρ over a field \mathbb{F} possesses an $\rho \times \rho$ submatrix of constant rank ρ if and only if $|\mathbb{F}| \geq \rho$.*

They showed that if $|\mathbb{F}| < \rho$ then there exist examples of partial matrices of size $m \times n$ with $\max\{m, n\} \geq \rho + |\mathbb{F}| - 1$ that does not contain a $\rho \times \rho$ submatrix with rank ρ . For $\rho = 3$ and \mathbb{F}_2 they provided the following 4×3 partial matrix

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & x_3 \\ x_1 & 0 & 1 \\ 0 & x_2 & 1 \end{bmatrix} \quad (8)$$

that has constant rank 3 and has no 3×3 submatrix of constant rank 3.

Now we will state some definitions motivated by the previous remarks. Since partial matrices are a subclass of ACI-matrices, these definitions will be stated in the more general framework.

Definition 3.2. *Let A be a $m \times n$ ACI-matrix of constant rank ρ over a field \mathbb{F} . We say that:*

1. A is **row reducible** if it contains some row R such that the $(m - 1) \times n$ ACI-matrix obtained by deleting R from A has constant rank ρ . And A is **row irreducible** otherwise.
2. A is **column reducible** if it contains some column C such that the $m \times (n - 1)$ ACI-matrix obtained by deleting C from A has constant rank ρ . And A is **column irreducible** otherwise.
3. A is **reducible** if it is row reducible and/or column reducible. And A is **irreducible** otherwise.

With this terminology the partial matrix P given in (8) is irreducible. If we consider P as an ACI-matrix we might expect to find, in the equivalence class of P , some reducible ACI-matrix. That is, some ACI-matrix with a 3×3 ACI-submatrix of constant rank 3. But this is not the case.

Nevertheless, there are irreducible partial matrices such that its equivalence class contains reducible ACI-matrices. For instance, over \mathbb{F}_2 consider

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ x_1 & 1 & 0 & 0 & 0 \\ 1 & x_2 & 0 & 0 & 0 \\ 1 & 1 & x_3 & 1 & 0 \\ 1 & 0 & 0 & x_4 & 0 \\ 1 & 0 & 0 & 1 & x_5 \end{bmatrix} \xrightarrow[\sim]{\substack{\text{row}_4 \rightarrow \text{row}_4 + \text{row}_1 \\ \text{row}_7 \rightarrow \text{row}_7 + \text{row}_1}} F = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ x_1 & 1 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 & 1 \\ 1 & 1 & x_3 & 1 & 0 \\ 1 & 0 & 0 & x_4 & 0 \\ 0 & 0 & 0 & 1 & x_5 + 1 \end{bmatrix}. \quad (9)$$

It can be checked that the partial matrix E is irreducible and has constant rank 5, that F is equivalent to E and so has constant rank 5, and that F is row reducible: if we delete its first row we obtain an ACI-matrix of constant rank 5.²

As we explained in Section 1.3, equivalent ACI-matrices represent the same geometrical collection of objects. So it would make sense to have a stronger concept of irreducibility, one that is preserved by equivalence. This motivates the following definition.

Definition 3.3. Let A be a $m \times n$ ACI-matrix of constant rank ρ over a field \mathbb{F} . We say that A is **completely irreducible** if each ACI-matrix equivalent to A is irreducible.

We have studied the effect on the rank of an ACI-matrix of constant rank when we delete one of its columns (or one of its rows). We are also interested in the effect on the rank of an ACI-matrix of constant rank when we add one constant column.

Definition 3.4. Let A be a $m \times n$ ACI-matrix of constant rank ρ over a field \mathbb{F} . We say that A is **column augmentable** if there exists some $v \in \mathbb{F}^m$ such that the augmented ACI-matrix $[A \ v]$ is of constant rank $\rho + 1$. Otherwise we will say that A is **column non-augmentable**.

Remark 3.5. Let A be a $m \times n$ ACI-matrix of constant rank ρ . From Definitions 1.2, 3.2 and 3.4 it follows that:

- (1) A is minimal full rank if and only if $\rho = m < n$ and A is column irreducible.
- (2) A is maximal full rank if and only if $\rho = n < m$ and A is column non-augmentable.

In the next result we will see how we can study the complete irreducibility of an ACI-matrix without considering all its equivalent ACI-matrices.

Theorem 3.6. The $m \times n$ ACI-matrix A of constant rank ρ is completely irreducible if and only if:

- (a) A is column irreducible.
- (b) A is column non-augmentable.

Proof. That A is completely irreducible means that TAQ is irreducible for any nonsingular constant T of order m and any permutation Q of order n or, equivalently, that TA is irreducible for any nonsingular constant T of order m . In turn, this is equal to say that TA is column irreducible and row irreducible for any nonsingular constant T of order m . And observe that TA is column irreducible if and only if A is column irreducible because

$$\text{rank}(C_j(TA)) = \text{rank}(TC_j(A)) = \text{rank}(C_j(A))$$

where $C_j(A)$ and $C_j(TA)$ denote the ACI-matrices obtained by deleting the column j of A and TA .

In summary, A is completely irreducible if and only if A is column irreducible and TA is row irreducible for any nonsingular constant T of order m . Observe that we finish the proof of our theorem if we prove that:

²If we do not impose to the irreducible matrix to be partial then there are much simpler examples than (9). Consider for instance the two equivalent ACI-matrices $E' = \begin{bmatrix} x \\ 1+x \end{bmatrix} \sim F' = \begin{bmatrix} x \\ 1 \end{bmatrix}$ over any field: E' is irreducible of constant rank one, F' is equivalent to E' , and F' is row reducible.

TA is row irreducible for any nonsingular constant $T \iff A$ is column non-augmentable.

Actually we will prove the opposite affirmation:

TA is row reducible for some nonsingular constant $T \iff A$ is column augmentable.

\Rightarrow) Let T be a nonsingular constant matrix of order m such that TA be row reducible. Then there is an $i \in \{1, \dots, m\}$ such that if we delete the i -th row from TA the resulting ACI-matrix remains of constant rank ρ . Without loss of generality assume that $i = 1$. Let $R_1(TA)$ be the ACI-matrix that we obtain deleting the first row of TA . Then

$$\text{rank}[TA|e_1] = \text{rank} \left[TA \left| \begin{array}{c} 1 \\ 0 \\ \vdots \\ 0 \end{array} \right. \right] = \text{rank} \left[\begin{array}{c|c} * & \begin{array}{c} 1 \\ 0 \\ \vdots \\ 0 \end{array} \\ \hline R_1(TA) & \begin{array}{c} 0 \\ \vdots \\ 0 \end{array} \end{array} \right] = \{\rho + 1\}$$

so

$$\text{rank}[A|T^{-1}e_1] = \text{rank}(T[A|T^{-1}e_1]) = \text{rank}[TA|e_1] = \{\rho + 1\}.$$

Therefore A is column augmentable with vector $T^{-1}e_1$.

\Leftarrow) Let v be a non-zero constant vector for which $[A|v]$ has constant rank $\rho + 1$. Let T be a nonsingular constant matrix of order m such that $Tv = e_1$. Then

$$\text{rank} \left[\begin{array}{c|c} * & \begin{array}{c} 1 \\ 0 \\ \vdots \\ 0 \end{array} \\ \hline R_1(TA) & \begin{array}{c} 0 \\ \vdots \\ 0 \end{array} \end{array} \right] = \text{rank}[TA|e_1] = \text{rank}(T^{-1}[TA|e_1]) = \text{rank}[A|v] = \{\rho + 1\}.$$

So $R_1(TA)$ has constant rank ρ , and so TA is row reducible because of its first row.

□

4 Completely irreducible ACI-matrices

The previous section should have convinced us that completely irreducible ACI-matrices deserve to be analyzed and fully understood. We will first analyze completely irreducible ACI-matrices which are full rank, after that those which are not full rank. Then we will make some remarks on how to construct completely irreducible ACI-matrices. And finally we will establish where do the completely irreducible ACI-matrices appear in Theorem 2.5.

4.1 Completely irreducible ACI-matrices which are full rank

In the next result we will show that the concept of complete irreducibility for full rank ACI-matrices encompasses the concepts of square, minimal and maximal full rank.

Proposition 4.1. *Let A be an $m \times n$ ACI-matrix of constant rank ρ . Then*

- (i) *A is completely irreducible of constant rank $\rho = m = n$ if and only if A is square full rank.*
- (ii) *A is completely irreducible of constant rank $\rho = m < n$ if and only if A is minimal full rank.*
- (iii) *A is completely irreducible of constant rank $\rho = n < m$ if and only if A is maximal full rank.*

Proof. (i) The necessary part is trivial. The sufficient part is based in two clear facts: that a square full rank ACI-matrix is irreducible, and that the ACI-matrices which are equivalent to a square full rank ACI-matrix are square full rank.

(ii) \Rightarrow) Assume that A is completely irreducible of constant rank $\rho = m < n$. By item (a) of Theorem 3.6, A is column irreducible. So, by item (1) of Remark 3.5, A is minimal full rank.

\Leftarrow) Assume that A is minimal full rank. So A has constant rank $m < n$ and therefore A is column non-augmentable, since for each $v \in \mathbb{F}^m$ the augmented matrix $[A|v]$ has size $m \times (n + 1)$ and constant rank m . On the other hand, by item (1) of Remark 3.5, A is column irreducible. So, by Theorem 3.6 A is completely irreducible of constant rank $m < n$.

(iii) \Rightarrow) Assume that A is completely irreducible of constant rank $\rho = n < m$. By item (b) of Theorem 3.6, A is column non-augmentable. So, by item (2) of Remark 3.5, A is maximal full rank.

\Leftarrow) Assume that A is maximal full rank. So A has constant rank $n < m$ and therefore A is column irreducible, since if we delete one column of A we obtain an ACI-matrix of size $m \times (n - 1)$ and constant rank $n - 1$. On the other hand, by item (2) of Remark 3.5, A is column non-augmentable. So, by Theorem 3.6 A is completely irreducible of constant rank $n < m$. \square

4.2 Completely irreducible ACI-matrices which are not full rank

In our next result we characterize the completely irreducible ACI-matrices which are not full rank.

Theorem 4.2. *Let A be a $m \times n$ ACI-matrix over a field \mathbb{F} and let ρ an integer with $1 \leq \rho < \min\{m, n\}$. Then A is completely irreducible of constant rank ρ if and only if for some positive integers ρ_1 and ρ_2 such that $\rho_1 + \rho_2 = \rho$ we have that $A \sim \begin{bmatrix} A_{11} & * \\ 0 & A_{22} \end{bmatrix}$ where A_{11} is minimal full rank of constant rank ρ_1 and A_{22} is maximal full rank of constant rank ρ_2 .*

Proof. Let A be a $m \times n$ ACI-matrix and let ρ an integer with $1 \leq \rho < \min\{m, n\}$.

\Rightarrow) As A has constant rank ρ with $1 \leq \rho < \min\{m, n\}$ then we can apply to A the item (iv) of Theorem 2.5. Moreover, by hypothesis A is completely irreducible and so we must apply exactly the case (a) of item (iv) of Theorem 2.5. So, for some positive integers $r < m$ and $s < n$ with $\rho = (m - r) + (n - s)$ there exist a nonsingular constant matrix T of order m and a permutation matrix Q of order n such that

$$TAQ = \left[\begin{array}{c|c} B & * \\ \hline 0_{r \times s} & C \end{array} \right]$$

with either $B = \begin{bmatrix} 1 & * \\ & \ddots \\ 0 & 1 \end{bmatrix}$ or B minimal full rank, and with either $C = \begin{bmatrix} 1 & * \\ & \ddots \\ 0 & 1 \end{bmatrix}$ or C maximal full rank. Let $\rho_1 = m - r$ and $\rho_2 = n - s$. Note that $\rho_1 > 0$, that $\rho_2 > 0$, and that $\rho_1 + \rho_2 = \rho$. As B has ρ_1 rows and C has ρ_2 columns then it follows from Lemma 2.1 that

$$\text{rank}(A) = \text{rank}(TAQ) = \{\rho_1 + \rho_2\} = \{\rho\}.$$

As A is completely irreducible then TAQ is irreducible and so

$$\left[\begin{array}{c|c} B & * \\ \hline 0_{r \times s} & C \end{array} \right] = \left[\begin{array}{c|c} B & * \\ \hline 0_{r \times s} & C \end{array} \right],$$

otherwise we could delete one row or one column without changing the rank of TAQ . Note that if $B = \begin{bmatrix} 1 & * \\ & \ddots \\ 0 & 1 \end{bmatrix}$ then $\text{rank}(A) = \{n\}$ and that if $C = \begin{bmatrix} 1 & * \\ & \ddots \\ 0 & 1 \end{bmatrix}$ then $\text{rank}(A) = \{m\}$. None of both possibilities are valid since A has constant rank $\rho < \min\{m, n\}$. Then B is minimal full rank of constant rank ρ_1 and C is maximal full rank of constant rank ρ_2 .

\Leftarrow) As complete irreducibility is preserved by equivalence then, without loss of generality, we can assume that $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$ where A_{11} is $\rho_1 \times (n - \rho_2)$ minimal full rank of constant rank ρ_1 and A_{22} is $(m - \rho_1) \times \rho_2$ maximal full rank of constant rank ρ_2 . From Theorem 3.6, we need to prove that:

– **A is column irreducible.**

We consider two cases. In both cases we will use the notation $C_k(H)$ for the ACI-matrix obtained by deleting the column k of the ACI-matrix H .

- 1) $j \in \{1, \dots, n - \rho_2\}$. As A_{11} is minimal full rank then $\min\{\text{rank}(C_j(A_{11}))\} < \rho_1$. So
- $$\min\{\text{rank}(C_j(A))\} = \min\left\{\text{rank}\begin{bmatrix} C_j(A_{11}) & A_{12} \\ 0 & A_{22} \end{bmatrix}\right\} = \min\left\{\text{rank}\begin{bmatrix} C_j(A_{11}) \\ 0 \end{bmatrix}\right\} + \rho_2 = (\rho_1 - 1) + \rho_2 = \rho - 1.$$
- 2) $j \in \{n - \rho_2 + 1, \dots, n\}$. As A_{22} is maximal full rank then $\text{rank}(C_{j-(n-\rho_2)}(A_{22})) = \{\rho_2 - 1\}$. So we can apply Lemma 2.1 to conclude that

$$\text{rank}(C_j(A)) = \text{rank}\begin{bmatrix} A_{11} & C_{j-(n-\rho_2)}(A_{12}) \\ 0 & C_{j-(n-\rho_2)}(A_{22}) \end{bmatrix} = \{\rho_1 + (\rho_2 - 1)\} = \{\rho - 1\}.$$

From 1) and 2) we conclude that A is column irreducible.

– $\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{array}{c} u_1 \\ u_2 \end{array}$ **is never of constant rank $\rho + 1$ for $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \mathbb{F}^m$.**

Since A_{22} is maximal full rank then $[A_{22} \mid u_2]$ is not full rank. So $[A_{22} \mid u_2]$ has a completion $[\widehat{A_{22}} \mid u_2]$ for which there exists a nonzero constant vector $\begin{bmatrix} w \\ \lambda \end{bmatrix} \in \mathbb{F}^{\rho_2+1}$ with $\lambda \neq 0$ such that:

$$[\widehat{A_{22}} \mid u_2] \begin{bmatrix} w \\ \lambda \end{bmatrix} = \mathbf{0} \in \mathbb{F}^{m-\rho_1}.$$

As $[A_{22} \mid u_2]$ and $[A_{12} \mid u_1]$ may share variables then the completion of $[A_{22} \mid u_2]$ may force a partial completion of $[A_{12} \mid u_1]$ which we fully complete in any way we want to $[\widehat{A_{12}} \mid u_1]$. Define the constant vector

$$a := [\widehat{A_{12}} \mid u_1] \begin{bmatrix} w \\ \lambda \end{bmatrix} \in \mathbb{F}^{\rho_1}.$$

Since A_{11} is full rank, then for any completion $\widehat{A_{11}}$ there exists a constant vector $v \in \mathbb{F}^{n-\rho_2}$ such that $\widehat{A_{11}} v = -a$.

Finally,

$$\begin{bmatrix} \widehat{A_{11}} & \widehat{A_{12}} \\ 0 & \widehat{A_{22}} \end{bmatrix} \begin{array}{c} u_1 \\ u_2 \end{array} \begin{bmatrix} v \\ w \\ \lambda \end{bmatrix} = \begin{bmatrix} -a + a \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} \in \mathbb{F}^m. \quad (10)$$

Note that $\lambda \neq 0$. Therefore $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ depends linearly of the columns of $\begin{bmatrix} \widehat{A_{11}} & \widehat{A_{12}} \\ 0 & \widehat{A_{22}} \end{bmatrix}$, thus the completion $\begin{bmatrix} \widehat{A_{11}} & \widehat{A_{12}} \\ 0 & \widehat{A_{22}} \end{bmatrix} \begin{array}{c} u_1 \\ u_2 \end{array}$ has rank ρ .

□

4.3 Constructing Completely Irreducible ACI-matrices

In Theorem 4.2 we have seen that if $A = \begin{bmatrix} A_{11} & * \\ 0 & A_{22} \end{bmatrix}$ where A_{11} is minimal full rank and A_{22} is maximal full rank then A is completely irreducible. This takes us to consider the question of whether we can use the completely irreducible ACI-matrices which are full rank (square, minimal and maximal) as building blocks to construct new completely irreducible ACI-matrices.

Namely, the question can be stated in the following terms: Is the ACI-matrix

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

completely irreducible where A_{11} and A_{22} are either square, minimal or maximal full rank? We have nine different cases:

(i) If A_{11} and A_{22} are square full rank, then A is completely irreducible.

Clearly A is square full rank and, by Proposition 4.1, it is completely irreducible.

(ii) If A_{11} is square full rank and A_{22} is minimal full rank, then A is not always completely irreducible.

Consider the ACI-matrix over \mathbb{F}_2

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} = \left[\begin{array}{cc|ccccc} 1 & x_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & y_1 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 1 & y_2 & y_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & y_3 & y_4 & 1 \\ 0 & 0 & y_1 & 1 & 1 & 1 & y_5 \end{array} \right].$$

It satisfies the following facts:

- (a) A_{11} is square full rank.
- (b) A_{22} is minimal full rank. First, that A_{22} is full rank can be checked directly since it has 5 variables and admits $32 = 2^5$ different completions, all of them of rank 3. Moreover, if we delete any of its five columns the resulting 3×4 ACI-matrix is not full rank since it admits a completion of rank 2.
- (c) A is full rank. It follows from the structure of A since any completion of A_{11} has rank 2 and any completion of A_{22} has rank 3, so any completion of A has rank 5.
- (d) A is column reducible. If we delete the second column of A then we obtain an ACI-matrix A' that has 5 variables and admits $32 = 2^5$ different completions, all of them of rank 5. So A' is full rank, which implies that A is column reducible.

Since A is column reducible then A is not irreducible, and thus A is not completely irreducible.

(iii) If A_{11} is square full rank and A_{22} is maximal full rank, then A is completely irreducible.

That A is maximal full rank was proved in [2, Lemma 2.2], and that A is completely irreducible is a consequence of Proposition 4.1.

(iv) If A_{11} is minimal full rank and A_{22} is square full rank, then A is completely irreducible.

Consider the $m \times n$ ACI-matrix

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

where A_{11} is $\rho_1 \times \theta_1$ minimal full rank (so it has constant rank ρ_1 and $\rho_1 < \theta_1$) and A_{22} is $\rho_2 \times \rho_2$ square full rank (so it has constant rank ρ_2). By Applying Lemma 2.1 we have that A is full rank of constant rank $\rho_1 + \rho_2 = m$. Observe also that $m = \rho_1 + \rho_2 < \theta_1 + \rho_2 = n$.

According to Proposition 4.1 the result follows if we prove that A is minimal full rank. And in turn, according to Remark 3.5 (1), it is enough to show that the ACI-matrix obtained by deleting any column of A admits a completion of rank $m - 1$:

- If we delete one of its first θ_1 columns then the resulting ACI-matrix is of type $\begin{bmatrix} A'_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$.

As A_{11} is minimal full rank, then there exists a completion $\widehat{A'_{11}}$ of A'_{11} whose rank is $\rho_1 - 1$.

Extend this completion so that $\begin{bmatrix} \widehat{A'_{11}} & \widehat{A_{12}} \\ 0 & \widehat{A_{22}} \end{bmatrix}$ is a completion of $\begin{bmatrix} A'_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$. Then

$$\text{rank} \begin{bmatrix} \widehat{A'_{11}} & \widehat{A_{12}} \\ 0 & \widehat{A_{22}} \end{bmatrix} = \text{rank} \begin{bmatrix} \widehat{A'_{11}} & 0 \\ 0 & \widehat{A_{22}} \end{bmatrix} = \text{rank}(\widehat{A'_{11}}) + \text{rank}(\widehat{A_{22}}) = (\rho_1 - 1) + \rho_2 = m - 1$$

- If we delete one of its last ρ_2 columns then the resulting ACI-matrix is of type $\begin{bmatrix} A_{11} & A'_{12} \\ 0 & A'_{22} \end{bmatrix}$.

The $\rho_2 \times (\rho_2 - 1)$ ACI-matrix A'_{22} has constant rank $\rho_2 - 1$. By Lemma 2.1 we have that $\begin{bmatrix} A_{11} & A'_{12} \\ 0 & A'_{22} \end{bmatrix}$ has constant rank $\rho_1 + (\rho_2 - 1) = m - 1$.

(v) If A_{11} and A_{22} are minimal full rank, then A is not always completely irreducible.

Consider the ACI-matrix over \mathbb{F}_2

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} = \left[\begin{array}{ccc|ccccc} x_1 & 1 & x_3 & 0 & 0 & 0 & 0 & 0 \\ x_1 + 1 & x_2 & 1 & y_1 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 & y_2 & y_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & y_3 & y_4 & 1 \\ 0 & 0 & 0 & y_1 & 1 & 1 & 1 & y_5 \end{array} \right]$$

We can check that A is full rank since any completion has rank 5, that A_{11} and A_{22} are minimal full rank, and that A is column reducible since the ACI-matrix obtained after deleting the first column of A has constant rank 5. The checks are similar to those on item (ii). So A is not completely irreducible.

(vi) If A_{11} is minimal full rank and A_{22} is maximal full rank, then A is completely irreducible.

This is part of Theorem 4.2 and it is quite surprising. Observe that Theorem 4.2 tell us that, up to equivalence, every completely irreducible ACI-matrices which is non-full rank can be constructed in this way.

(vii) If A_{11} is maximal full rank and A_{22} is square full rank, then A is completely irreducible.

In this case the position of A_{11} and A_{22} is interchanged with respect to the case (iii) above. We can adapt easily the proof in [2, Lemma 2.2] to show that A is maximal full rank. And that A is completely irreducible is a consequence of Proposition 4.1.

(viii) If A_{11} is maximal full rank and A_{22} is minimal full rank, then A is not always completely irreducible.

Consider the ACI-matrix over \mathbb{F}_2

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & x_1 & 0 & 0 & 0 \\ x_2 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & x_3 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & y_1 & 1 & y_3 \\ 0 & 0 & 0 & y_1 + 1 & y_2 & 1 \end{array} \right]$$

It can be checked that A_{11} is maximal full rank, that A_{22} is minimal full rank, and that some completions of A have rank 5 and other completions of A have rank 6. Then A has no constant rank. So it can not be completely irreducible.

(ix) If A_{11} and A_{22} are maximal full rank, then A is completely irreducible.

That A is maximal full rank was proved in [2, Lemma 2.3], and that A is completely irreducible is a consequence of Proposition 4.1.

The following table summarizes all the possibilities:

A_{11}			$\left. \begin{array}{l} \text{Square FR} \\ \text{Minimal FR} \\ \text{Maximal FR} \end{array} \right\} A_{22} \quad (11)$
Square FR	Minimal FR	Maximal FR	
(i) C.I.	(iv) C.I.	(vii) C.I.	
(ii) Not always C.I.	(v) Not always C.I.	(viii) Not always C.I.	
(iii) C.I.	(vi) C.I.	(ix) C.I.	

5 The core of a constant rank ACI-matrix

We know that not all ACI-matrices of constant rank ρ have a $\rho \times \rho$ submatrix of constant rank ρ . It could be expected that each ACI-matrix of constant rank ρ has at least a submatrix of constant rank ρ that is completely irreducible. But again this is not the case. Consider the 7×5 partial matrix E given in (9): the unique submatrix of E of constant rank 5 is E , and E is not completely irreducible since $E \sim F$ and F is row reducible.

So not all ACI-matrices of constant rank ρ have a submatrix of constant rank ρ that is completely irreducible. This removes one tool that could be employed in the calculus of the rank of an ACI-matrix. We can in some way offset this situation.

Definition 5.1. *Let A be an ACI-matrix of constant rank ρ over a field \mathbb{F} . If*

$$A \sim \begin{bmatrix} A' & * \\ * & * \end{bmatrix}$$

*where A' is completely irreducible of constant rank ρ then A' is said to be a **core** of A .*

So, complete irreducibility allows to generalize, in some way, the concept of $\rho \times \rho$ submatrix of rank ρ . Given an ACI-matrix A with constant rank ρ it seems like a good idea to find a representative of its equivalence class that verifies that: it has a completely irreducible ACI-submatrix of rank ρ (a core), and has a structure which is simple and makes it clear why it is of constant rank ρ . Theorem 2.5 finds such a representative, as we will see in the proof of the next result.

Lemma 5.2. *Any ACI-matrix of constant rank over a field has a core.*

Proof. Let A be a $m \times n$ ACI-matrix with constant rank ρ . We have several possibilities:

(i) If $\rho = m = n$ then, by Proposition 4.1, A is completely irreducible. So A is a core of A .

We can find a core of A with a simpler structure. According to Theorem 2.5, $A \sim \begin{bmatrix} 1 & \cdots & * \\ 0 & & 1 \end{bmatrix}$. So $\begin{bmatrix} 1 & \cdots & * \\ 0 & & 1 \end{bmatrix}$, which is completely irreducible, is also a core of A .

(ii) If $\rho = m < n$ then, according to Theorem 2.5, $A \sim [B \quad *]$ where $B = \begin{bmatrix} 1 & \cdots & * \\ 0 & & 1 \end{bmatrix}$ of size $m \times m$ or B is minimal full rank of size $m \times n'$ with $m < n' \leq n$. In any case B is completely irreducible with constant rank ρ (see Proposition 4.1). So B is a core of A .

(iii) If $\rho = n < m$ then, according to Theorem 2.5, $A \sim \begin{bmatrix} * \\ C \end{bmatrix}$ where $C = \begin{bmatrix} 1 & \cdots & * \\ 0 & & 1 \end{bmatrix}$ of size $n \times n$ or C is maximal full rank of size $m' \times n$ with $n < m' \leq m$. In any case C is completely irreducible with constant rank ρ (see Proposition 4.1). So C is a core of A .

(iv) If $\rho < \min\{m, n\}$ then, according to Theorem 2.5, we have three possibilities:

(a) For some positive integers $r < m$ and $s < n$ with $\rho = (m - r) + (n - s)$ we have

$$A \sim \left[\begin{array}{cc|c} B & * & * \\ \hline 0_{r \times s} & & C \end{array} \right]$$

where $B = \begin{bmatrix} 1 & \cdots & * \\ 0 & & 1 \end{bmatrix}$ or B is minimal full rank and $C = \begin{bmatrix} 1 & \cdots & * \\ 0 & & 1 \end{bmatrix}$ or C is maximal full rank. By permuting some rows and some columns of the last ACI-matrix we have that

$$\left[\begin{array}{cc|c} B & * & * \\ \hline 0_{r \times s} & & C \end{array} \right] \sim \left[\begin{array}{cc|c} B & * & * \\ 0 & C & 0 \\ \hline 0 & * & 0 \end{array} \right]$$

According to the table given in (11) the ACI-matrix $\begin{bmatrix} B & * \\ 0 & C \end{bmatrix}$ is completely irreducible (the possible cases are those corresponding to items (i), (iii), (iv) or (vi)) with constant rank $(m - r) + (n - s)$ (see Lemma 2.1), then it is a core of A .

- (b) For some positive integer $r < m$ with $\rho = m - r$ we have $A \sim \left[\begin{array}{c|c} B & * \\ \hline 0_{r \times n} & \end{array} \right]$ where $B = \begin{bmatrix} 1 & \cdots & * \\ 0 & & 1 \end{bmatrix}$ or B is minimal full rank. As B is completely irreducible with constant rank $m - r$ then B is a core of A .
- (c) For some positive integer $s < n$ with $\rho = n - s$ we have $A \sim \left[\begin{array}{c|c} & * \\ \hline 0_{m \times s} & C \end{array} \right]$ where $C = \begin{bmatrix} 1 & \cdots & * \\ 0 & & 1 \end{bmatrix}$ or C is maximal full rank. As C is completely irreducible with constant rank $n - s$ then C is a core of A .

□

Observe that Lemma 5.2 for ACI-matrices of constant rank has analogy with Theorem 3.1 for partial matrices of constant rank.

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